Application of the rule for the multiplication of determinants gives:

$$Z^2 = \begin{vmatrix} Q - y^2 & -yx & -yv \\ -xy & Q - x^2 & -xv \\ -vy & -vx & Q - v^2 \end{vmatrix}.$$

Dividing the first, second, and third *columns* by y, x, v, respectively, and *multiplying* the first, second, and third *rows*, respectively, by the same symbols gives:

$$Z^2 = \begin{vmatrix} Q - y^2 & -y^2 & -y^2 \\ -x^2 & Q - x^2 & -x^2 \\ -v^2 & -v^2 & Q - v^2 \end{vmatrix},$$

which easily reduces to $Z^2 = Q^2(Q - v^2 - y^2 - x^2) = Q^2z^2$.

In exactly the same way, $V^2 = Q^2v^2$, $X^2 = Q^2x^2$, $Y^2 = Q^2y^2$. Whence

$$\{Z^2 + V^2 + X^2 + Y^2\} = Q^2(z^2 + v^2 + x^2 + y^2) = Q^3.$$

Direct application of the rule for the multiplication of determinants gives for H^2 a determinant wherein each element in the leading diagonal is Q, and all the other elements are zero; whence $H^2 = Q^4$. These values set in the equation prove it to be correct.

Also solved by J. L. Riley, E. H. Worthington, A. M. Harding, J. B. Rosenbach, and the Proposer.

GEOMETRY.

511. Proposed by FRANK V. MORLEY, Student, Haverford, College.

Let a_i $(i = 1, 2, 3, 4)$ be four points on a circle, and let the incenter of the triangle formed by omitting a_i be c_i; prove that the four points c_i form a rectangle.

Solution by the Proposer.

Let there be four circles with centers, m_{12}, m_{23}, m_{34}, m_{41}, on the given circle, and let each circle intersect the next in two points, of which one set, a_2, a_3, a_4, a_1, lie on the given circle, and one set, c_4, c_1, c_2, c_3, lie inside, as shown in Fig. 1.

Then $\measuredangle\, c_4a_1a_2 = \frac{1}{2}\measuredangle\, c_4m_{12}a_2$, and $\measuredangle\, m_{23}a_1a_2 = \frac{1}{2}a_3m_{12}a_2$ (1).

$\therefore \measuredangle\, c_4c_3a_2 = \measuredangle\, c_4a_1a_2 = \measuredangle\, m_{23}a_1a_2 = \measuredangle\, m_{23}m_{41}a_2$, and $c_3c_4 \,||\, m_{41}m_{23}$.

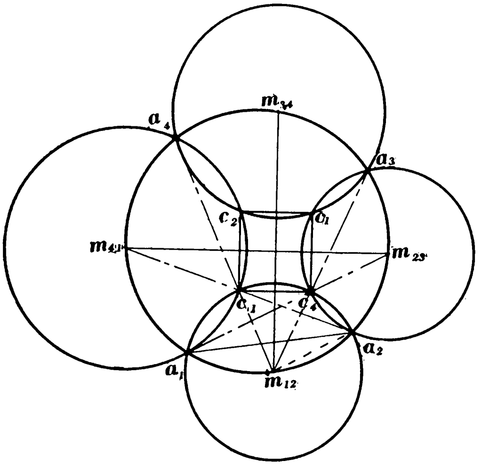

Fig. 1.

By equality of arcs, $m_{41}m_{23} \perp m_{12}m_{34}$, and hence $c_3c_4 \perp m_{12}m_{34}$.

Since $\measuredangle\, a_4m_{12}a_3 \equiv \measuredangle\, c_3m_{12}c_4$, and is bisected by $m_{12}m_{34}$, c_3, c_4 are reflections through $m_{12}m_{34}$, and likewise c_1, c_2; similarly c_1, c_4, and c_2, c_3, are reflections through $m_{41}m_{23}$.

$\therefore c_1, c_2, c_3, c_4$, form a rectangle.

But by (1), c_4 is the intersection of a_3m_{12}, a_1m_{23}, or the incenter of $a_1a_2a_3$. Therefore c_i as defined above is identical with the incenter of the triangle formed by omitting a_i, and the problem is proved.

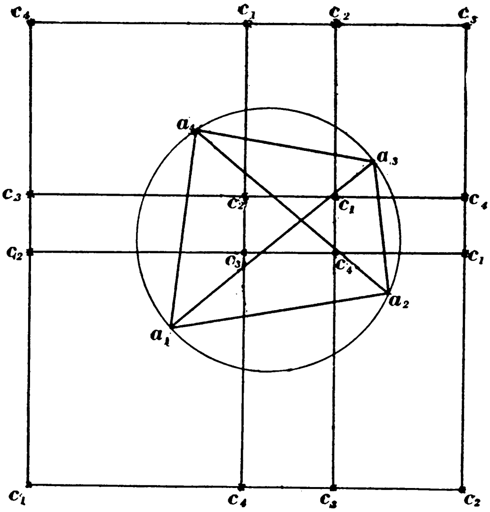

FIG. 2.

The extension suggested to be proved by analogous methods is shown in Fig. 2, where the four incenters and twelve excenters, of the four triangles formed by omitting a_i in turn, are the intersections of two sets of four perpendicular lines.

Also solved by O. J. RAMLER, R. A. JOHNSON, J. E. ROWE, J. W. CLAWSON, and PAUL CAPRON.

CALCULUS.

409. Proposed by B. J. BROWN, Victor, Colorado.

Integrate the differential equation,

$$\frac{\partial^2 z}{\partial x \partial y} + \frac{1}{x+y}\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) - \frac{2}{(x+y)^2} z = 0.$$

II. SOLUTION BY W. W. BEMAN, Ann Arbor, Michigan.

This problem is found in Gregory's *Examples* and Forsyth's *Treatise on Differential Equations*. Gregory puts

$$y + x = u, \qquad y - x = v$$

and makes use of the solution of a previous problem.

Maser in the German translation of Forsyth's *Treatise* puts

$$z = \frac{u}{(x+y)^2} \text{ and writes the new equation } (x+y)\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.$$

Differentiating with respect to x, he gets

$$(x+y)\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^2 u}{\partial x^2}, \text{ and from this } \frac{\partial^2 u}{\partial x^2} = (x+y)\phi'''(x).$$

Integrating twice with respect to x, he obtains u and finally z as in Mr. Adams's solution on p. 129 of the March, 1917, MONTHLY. It may be observed that in Mr. Adams's solution

$$w = \frac{\partial^2 u}{\partial x^2}.$$